

Ergodicity and hydrodynamic limits for an epidemic model

Lamia Belhadji

Abstract We consider two approaches to study the spread of infectious diseases within a spatially structured population distributed in social clusters. According whether we consider only the population of infected individuals or both populations of infected individuals and healthy ones, two models are given to study an epidemic phenomenon. Our first approach is at a microscopic level, its goal is to determine if an epidemic may occur for those models. The second one is the derivation of hydrodynamics limits. By using the relative entropy method we prove that the empirical measures of infected and healthy individuals converge to a deterministic measure absolutely continuous with respect to the Lebesgue measure, whose density is the solution of a system of reaction-diffusion equations.

1. Introduction

We study an epidemic model describing the course of a single disease within a spatially structured human population distributed in *social clusters of finite or infinite size*. That is, each site of the d -dimensional integer lattice \mathbb{Z}^d is occupied by a cluster of *individuals*, each individual can be *healthy* or *infected* and the number of infected individuals at each cluster is either bounded or may be infinite. A cluster is said to be infected if it contains at least one infected individual and is said to be healthy otherwise. The first model we investigate is an extension of a process introduced in Schinazi (2002), and will be referred to as the *cluster recovery process* (CRP). The second with another recovery mechanism extend a process introduced in Belhadji and Lanchier (2006), and will be referred to as the *individual recovery process* (IRP).

For both CRP (Schinazi, 2002) and IRP (Belhadji and Lanchier, 2006), the dynamics depends on three parameters, namely the outside infection rate λ (the rate at which an individual infects healthy individuals of other clusters), the within infection rate ϕ (the rate at which an individual infects healthy individuals present in the same cluster), and the cluster size κ (can be seen as the mean number of individuals having sustained contacts with a given individual). In both models, it is assumed that, once a cluster has at least one infected individual, infections within the cluster are a lot more likely than additional infections from the outside so we neglect the latter. The only difference between the CRP and the IRP is the recovery mechanism. For the CRP, all the infected individuals in a given cluster are simultaneously replaced by healthy individuals, which follows from the assumption that, once an infected individual is discovered, its social cluster rapidly recovers thanks to an antidote. For the IRP, we deal with the other extreme case, that is we assume that at most one infected individual recovers at once, that is the tracking system is not effective enough and the infection can spread within a given cluster before it is detected. In particular, the CRP and the IRP can be considered as spatial stochastic models for the transmission of infectious diseases in developed and developing countries, respectively.

We assume that individuals within the same cluster have repeated contacts whereas the individuals belonging to neighboring clusters have casual contacts only. This suggests that the infection spreads out faster within clusters than between them, this is the reason why we introduce in the IRP and CRP an other outside infection rate β (at which an individual infects healthy individuals of other infected clusters). This allows us to take β lower than ϕ , to favorate within infections. More general than the processes of Schinazi (2002) and Belhadji and Lanchier, we will assume that

AMS 2000 subject classifications: Primary 60K35; 82C22. Secondary 92D25.

Keywords and phrases: Infinite particle systems, contact process, invariant measures, hydrodynamic limits, epidemic model, coupling, reaction diffusion process.

an outside infection may occur even if the cluster is already infected, and to avoid the condition that the number of infected individuals is bounded by κ we will deal with IRP and CRP with infinite cluster size.

The first aim of this paper is to investigate the probability of an epidemic for both processes depending on the value of each of the three parameters λ , β and ϕ .

In both the cluster recovery process (Schinazi, 2002) and individual recovery process (Belhadji and Lanchier, 2006), only the population of infected individuals is taken into account; we will consider more general Markov processes evolving on the 1-dimensional lattice, without any restrictions on the clusters sizes, and with two types of particles, healthy and infected individuals.

In this model healthy individuals get infected with the same infection mechanism as in CRP, infected individuals recover at rate 1 and moreover individuals are born, die and migrate, the migration of individuals (infected or healthy) speeded up by renormalizing parameter N^2 . By using the relative entropy method, and in particular the works Mourragui (1996), Perrut (2000), we will prove that the process admits hydrodynamic limits, that is by rescaling space and time the densities of healthy and infected individuals evolve according to nonlinear reaction-diffusion equations.

2. Presentation of the models and results

In order to investigate the individual and cluster recoveries processes with infinite cluster size, we start by introducing the evolution of the individual and cluster recoveries processes with finite cluster size $\kappa \in \mathbb{N}$ denoted respectively by IRP(κ) and CRP(κ). The IRP(κ) is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \rightarrow \{0, 1, \dots, \kappa\} \subsetneq \mathbb{N}$, with κ denoting the common size of the clusters, and $\xi_t(x)$ indicates the number of infected individuals present in the cluster at time $t \geq 0$. To take into account the outside infections, we introduce an interaction neighborhood. For any $x, z \in \mathbb{Z}^d$, $x \sim z$ indicates that site z is one of the $2d$ nearest neighbors of site x . Let the transition probability: $p(x, y) = 1/2d \mathbf{1}_{\{\|x-y\|_1=1\}}$, where $\|x-y\|_1 = |x_1 - y_1| + \dots + |x_d - y_d|$. Then, the state of site x flips according to the transition rates:

$$0 \rightarrow 1 \quad \text{at rate} \quad 2d\lambda \sum_{z \in \mathbb{Z}^d} p(x, z) \xi(z) \quad (1)$$

$$i \rightarrow i+1 \quad \text{at rate} \quad 2d\beta \sum_{z \in \mathbb{Z}^d} p(x, z) \xi(z) + i\phi \quad i = 1, 2, \dots, \kappa-1 \quad (2)$$

$$i \rightarrow i-1 \quad \text{at rate} \quad i \quad i = 1, 2, \dots, \kappa. \quad (3)$$

That is, a healthy cluster at site x gets infected, i.e. the state of x flips from 0 to 1, at rate λ times the number of infected individuals present in the neighboring clusters. In other respects, if there are i infected individuals in the cluster x , $i = 1, 2, \dots, \kappa-1$, then the state of x flips from i to $i+1$ at rate β times the number of infected individuals present in the neighboring clusters plus $i\phi$ (each of infected individual infects healthy ones in the cluster x at rate ϕ). Finally, each infected individual recovers at rate 1 regardless of the number of infected individuals in its cluster.

The CRP(κ) is a Markov process $\eta_t : \mathbb{Z}^d \rightarrow \{0, 1, \dots, \kappa\}$, with $\eta_t(x)$ denoting the number of infected individuals at site x at time $t \geq 0$, and whose evolution is obtained by replacing the transition (3) above by

$$i \rightarrow 0 \quad \text{at rate} \quad 1 \quad i = 1, 2, \dots, \kappa. \quad (4)$$

That is, all the infected individuals in a given cluster are now simultaneously replaced by healthy ones at rate 1, the infection mechanism modelled by (1) and (2) being unchanged.

The graphical representation

An argument of Harris (1972) assures us of the existence and uniqueness of the models $\text{IRP}(\kappa)$ and $\text{CRP}(\kappa)$, for all $\kappa \geq 1$ finite. For each $x, z \in \mathbb{Z}^d$ with $x \sim z$ and $i = 1, 2, \dots, \kappa$, we let $\{T_n^{x,z,i} : n \geq 1\}$ (respectively, $\{\tilde{T}_n^{x,z,i} : n \geq 1\}$) denote the arrival times of independent Poisson processes with rate λ (respectively, β). To take into account the within infections, we introduce, for $x \in \mathbb{Z}^d$ and $i = 1, 2, \dots, \kappa - 1$, a further collection of independent Poisson processes, denoted by $\{U_n^{x,i} : n \geq 1\}$, each of them has rate ϕ . Finally, for each $x \in \mathbb{Z}^d$ and $i = 1, 2, \dots, \kappa$, we let $\{V_n^{x,i} : n \geq 1\}$ be the arrival times of independent rate 1 Poisson processes.

Given initial configurations ξ_0 and η_0 , and the graphical representation introduced above, the process can be constructed as follows. If there are at least i infected individuals at site x at time $T_n^{x,z,i}$ (respectively, $\tilde{T}_n^{x,z,i}$), then if site z is in state $j = 0$ (respectively, $j, j = 1, \dots, \kappa - 1$) it flips to $j + 1$ for both processes. In other respects, if there are j infected individuals, where $i \leq j \leq \kappa - 1$, at site x at time $U_n^{x,i}$, then one more individual gets infected in the cluster, i.e., the state of x flips from j to $j + 1$, for both processes. Finally, if there are j infected individuals, $1 \leq j \leq \kappa$, at site x at time $V_n^{x,i}$, then the state of x flips from j to $j - 1$ if and only if $i \leq j$ for the process ξ_t , while flips from j to 0 if and only if $i = 1$ for the process η_t . In particular, \times_i 's, $i = 2, 3, \dots, \kappa$, have no effect on the process η_t .

The epidemic behavior of $\text{IRP}(\infty)$ and $\text{CRP}(\infty)$

Assume now that each cluster may contain an infinite number of individuals. The resulting processes are denoted by $\text{IRP}(\infty)$ and $\text{CRP}(\infty)$. The $\text{IRP}(\infty)$ (respectively, $\text{CRP}(\infty)$) is a continuous-time Markov process in which the state at time t is a function $\xi_t : \mathbb{Z}^d \rightarrow \mathbb{N}$, (respectively, $\eta_t : \mathbb{Z}^d \rightarrow \mathbb{N}$). The infection mechanism of the $\text{IRP}(\infty)$ and $\text{CRP}(\infty)$ is then described formally by setting $\kappa = \infty$ in the transitions (1), (2). In the same way, the recovery mechanism of the $\text{IRP}(\infty)$ (respectively, $\text{CRP}(\infty)$) is described by the transition (3) (respectively, (4)). To construct our processes we adopt an other point of view different from the graphical representation which moreover will allowed us to study their ergodicity. We rely on techniques introduced in Chen (1992) to prove the existence and uniqueness of the $\text{IRP}(\infty)$ and the $\text{CRP}(\infty)$ when $\beta \leq \lambda$.

We now discuss the effects of each of the three parameters, namely the outside infection rates λ and β , and the within infection rate ϕ , on the probability of an epidemic for both models.

From now on, we consider the processes starting with a single infected individual at site 0.

Definition 2.1 *We say that an epidemic may occur when*

$$P(\forall t \geq 0, \exists x \in \mathbb{Z}^d : \xi_t(x) \neq 0) > 0.$$

Otherwise, we say that there is no epidemic.

We prove by using basic coupling that the probability of an epidemic is nondecreasing with respect to the initial configuration and to each of the parameters λ , β , and ϕ .

Note that, when $\phi = 0$ and $\beta = 0$, there can be only one infected individual in each cluster so that both processes $\text{IRP}(\infty)$ and $\text{CRP}(\infty)$ are identical and reduce to the basic contact process with infection rate λ , in this case, there exists a critical value $\lambda_c \in (0, \infty)$ such that if $\lambda \leq \lambda_c$ then the processes converge in distribution to the “all 0” configuration; otherwise, an epidemic may occur. It follows by using basic coupling that an epidemic may occur whenever $\lambda > \lambda_c$ regardless of the value of the parameters ϕ and β and through a comparison with a branching random walk, we deduce that the processes $\text{IRP}(\infty)$ and $\text{CRP}(\infty)$ converge to the “all 0” configuration when $2d\lambda < 1$. When β or ϕ are different from 0, the limiting behavior of the process is more complicated to predict due to the combined effects of the three birth rates. We can however by using the ergodicity criterion introduced in Chen (1992) we extend the result in the following way.

Theorem 1 *If*

$$\phi + 2d\lambda < 1, \tag{5}$$

then there is no epidemic for the $IRP(\infty)$ and $CRP(\infty)$ with parameters (λ, β, ϕ) .

The cluster size being fixed, the analogue of Theorem 1 for $CRP(\kappa)$ and $IRP(\kappa)$ is given by:

Proposition 2.2 *If*

$$\phi + 2d(\lambda \vee \beta) < 1, \quad (6)$$

then there is no epidemic for the $IRP(\kappa)$ and the $CRP(\kappa)$ with parameters (λ, β, ϕ) for all $\kappa \geq 1$. Note that this condition is uniform in the cluster size.

The ergodicity criterion established in Theorem 1 shows that when (5) holds both $IRP(\infty)$ and $CRP(\infty)$ converge to the “all 0” configuration. Moreover, by using basic coupling, Theorem 1, Schinazi (2002) and Theorem 3, Belhadji and Ianchier (2006) we prove that when ϕ is large enough an epidemic may occur for the $IRP(\infty)$ and $CRP(\infty)$:

Theorem 2 *For all $\kappa \geq 2$, $\phi \geq 0$ and $\beta \geq 0$, if $\lambda > \lambda_c$ an epidemic may occur for the $IRP(\infty)$ and $CRP(\infty)$. For all λ and β with $\beta \leq \lambda < 1/2d$, there exists $\phi_c(\lambda, \beta) \in (0, \infty)$ such that: if $\phi < \phi_c(\lambda, \beta)$ there is no epidemic while an epidemic may occur for both processes if $\phi > \phi_c(\lambda, \beta)$.*

As a consequence of ergodicity criterion (6) and by analyzing the behavior of the processes $IRP(\kappa)$ and $CRP(\kappa)$ in the limiting case $\phi = \infty$, the analogues of Theorem 2 is given respectively by:

Proposition 2.3 *For all $\kappa \geq 2$, $\phi \geq 0$ and $\beta \geq 0$, if $\lambda > \lambda_c$ an epidemic may occur for the $IRP(\kappa)$. For all $\kappa \geq 2$, λ and β with $\lambda \vee \beta < 1/2d$, there exists $\phi_c(\lambda, \beta) \in (0, \infty)$ such that if $\phi < \phi_c(\lambda, \beta, \kappa)$ there is no epidemic, while if $\phi > \phi_c(\lambda, \beta, \kappa)$ an epidemic may occur for $IRP(\kappa)$.*

Proposition 2.4 *For all $\kappa \geq 1$, $\phi \geq 0$ and $\beta \geq 0$, if $\kappa\lambda \leq \lambda_c$ there is no epidemic while if $\lambda > \lambda_c$ an epidemic may occur for the $CRP(\kappa)$. For all $\kappa \geq 2$, $\lambda > \kappa\lambda_c$ and $\lambda \vee \beta < 1/2d$ there is $\phi_c(\lambda, \beta, \kappa) \in (0, \infty)$ such that if $\phi < \phi_c(\lambda, \beta, \kappa)$ there is no epidemic for the $CRP(\kappa)$ while if $\phi > \phi_c(\lambda, \beta, \kappa)$ an epidemic may occur.*

Hydrodynamic limits for a two-species IRP with infinite cluster size

In the previous models, only the population of infected individuals is taken into account; we consider now a more general Markov process, without any restrictions on the clusters sizes, and with two types of particles, healthy and infected individuals. This epidemic model is a continuous-time Markov process $(\eta_t, \xi_t)_{t \geq 0}$ in which the state at time t is a function $(\eta_t, \xi_t) : \mathbb{Z} \rightarrow \mathbb{N} \times \mathbb{N}$, where $\eta_t(x)$ and $\xi_t(x)$ are the respective numbers of healthy and infected individuals at site x and at time t . The dynamics splits into two parts: diffusion and reaction. The diffusion represents the migration of individuals (infected or healthy) speeded up by a renormalizing parameter N^2 , it consists in independent symmetric random walks with nearest neighbor jumps, accelerated by N^2 . There is an interaction between healthy and infected individuals in the reaction part, which describes births, deaths, recoveries and infections of individuals.

Our aim is to determine the limiting behavior of scaling processes as N goes to infinity, in others words we will prove hydrodynamic limits for this epidemic model. The strategy consists first in restricting the study to the torus then by coupling method to extend the result to all space.

To describe the evolution rules of the process, we set

$$\eta^{x,+}(z) = \begin{cases} \eta(z) + 1 & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x, \end{cases} \quad \text{and if } \eta(z) > 0, \quad \eta^{x,-}(z) = \begin{cases} \eta(z) - 1 & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x, \end{cases}$$

and

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

In other words, $\eta^{x,+}$ (respectively, $\eta^{x,-}$) is the configuration obtained from η by adding a particle at site x (respectively, removing a particle at site x if there is at least one). The configuration $\eta^{x,y}$ is obtained from η by letting one particle jump from x to y . The formal infinitesimal generator is given for a cylinder function f by

$$\Omega f(\eta, \xi) = \Omega^{\mathcal{R}} f(\eta, \xi) + N^2 \Omega^{\mathcal{D}} f(\eta, \xi) \quad (7)$$

where $\Omega^{\mathcal{D}} = \Omega^{\mathcal{D},1} + \Omega^{\mathcal{D},2}$, and $\Omega^{\mathcal{D},1}$ (respectively, $\Omega^{\mathcal{D},2}$) describes the migration of healthy (respectively, infected) individuals,

$$\Omega^{\mathcal{D},1} f(\eta, \xi) = \sum_{x,y \in \mathbb{Z}} \eta(x) p(x,y) \left[f(\eta^{x,y}, \xi) - f(\eta, \xi) \right] \quad (8)$$

$$\Omega^{\mathcal{D},2} f(\eta, \xi) = \sum_{x,y \in \mathbb{Z}} \xi(x) p(x,y) \left[f(\eta, \xi^{x,y}) - f(\eta, \xi) \right], \quad (9)$$

$p(x,y)$ is a transition probability on the lattice \mathbb{Z} such that a jump from site x to site y is allowed if and only if x and y are neighbors, given by $p(x,y) = \frac{1}{2} \mathbf{1}_{\{|x-y|=1\}}$, and

$$\begin{aligned} \Omega^{\mathcal{R}} f(\eta, \xi) &= \sum_{x \in \mathbb{Z}} \beta_1(\eta(x), \xi(x)) \left[f(\eta^{x,+}, \xi) - f(\eta, \xi) \right] + \delta_1(\eta(x), \xi(x)) \left[f(\eta^{x,-}, \xi) - f(\eta, \xi) \right] \\ &+ \sum_{x \in \mathbb{Z}} \beta_2(\eta(x), \xi(x)) \left[f(\eta, \xi^{x,+}) - f(\eta, \xi) \right] + \delta_2(\eta(x), \xi(x)) \left[f(\eta, \xi^{x,-}) - f(\eta, \xi) \right] \\ &+ \sum_{x \in \mathbb{Z}} \xi(x) \left[f(\eta^{x,+}, \xi^{x,-}) - f(\eta, \xi) \right] + \mathbf{1}_{\{\eta(x) > 0\}} \phi \xi(x) \left[f(\eta^{x,-}, \xi^{x,+}) - f(\eta, \xi) \right] \\ &+ \sum_{x \in \mathbb{Z}} \mathbf{1}_{\{\eta(x) > 0, \xi(x) = 0\}} \left(\lambda \sum_{y \sim x} \xi(y) \right) \left[f(\eta^{x,-}, \xi^{x,+}) - f(\eta, \xi) \right] \\ &+ \sum_{x \in \mathbb{Z}} \mathbf{1}_{\{\eta(x) > 0, \xi(x) > 0\}} \left(\beta \sum_{y \sim x} \xi(y) \right) \left[f(\eta^{x,-}, \xi^{x,+}) - f(\eta, \xi) \right], \end{aligned}$$

where

$$\begin{aligned} \beta_1(\eta(x), \xi(x)) &= \alpha_1 (\eta(x) + \xi(x)), & \delta_1(\eta(x), \xi(x)) &= \kappa \eta(x)^2 (\eta(x) + \xi(x)^2) \\ \beta_2(\eta(x), \xi(x)) &= \alpha_2 (\eta(x) + \xi(x)), & \delta_2(\eta(x), \xi(x)) &= \kappa \xi(x)^2 (\eta(x)^2 + \xi(x)), \end{aligned} \quad (10)$$

and α_1, α_2 and κ are positive coefficients. In other words, healthy (respectively, infected) individuals die at rate $\delta_1(\eta, \xi)$ (respectively, $\delta_2(\eta, \xi)$) and are born at rate $\beta_1(\eta, \xi)$ (respectively, $\beta_2(\eta, \xi)$); a healthy cluster at site x gets infected, that is the state of x flips from 0 to 1, at rate λ times the number of infected individuals present in the neighboring clusters. If there are $i \geq 1$ infected individuals in the cluster, then each of these individuals infects healthy individuals in the cluster at rate ϕ ; finally, each infected individual recovers at rate 1 regardless of the number of infected individuals in its cluster.

Theorems 13.8 and 13.18 in Chen 1992, enable to establish sufficient conditions for existence and uniqueness of the process $(\eta_t, \xi_t)_{t \in \mathbb{R}^+}$ whose evolution is described by the formal generator Ω in (7). We show that conditions called the first moment condition, Lipschitz conditions, growing condition and moment condition are satisfied for the process.

We first assume that healthy and infected individuals live on the space

$$\{x/N, x \in \mathbb{T}_N\}$$

where \mathbb{T}_N is the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ (i.e. sites 0 and $N-1$ are neighbors). We make the distance between two neighboring sites converging to zero by letting N goes to infinity. The evolution of the process is described by the generator

$$\Omega_N = \Omega_N^{\mathcal{R}} + N^2 \Omega_N^{\mathcal{D}}, \quad (11)$$

where $\Omega_N^{\mathcal{R}}$ and $\Omega_N^{\mathcal{D}}$ are the restrictions of $\Omega^{\mathcal{R}}$ and $\Omega^{\mathcal{D}}$ to \mathbb{T}_N . Let μ^N be the initial distribution of the process on $\mathbb{N}^{\mathbb{T}_N} \times \mathbb{N}^{\mathbb{T}_N}$ and S_t^N be the semi-group associated to the generator Ω_N . Using the relative entropy method (See Kipnis and Landim, 1999) we will prove that the empirical measure $(\pi_t^N(\eta_t), \pi_t^N(\xi_t))$, defined by

$$\pi_t^N(\eta_t) = \frac{1}{N} \sum_{x=0}^{N-1} \eta_t(x) \delta_{x/N}, \quad \pi_t^N(\xi_t) = \frac{1}{N} \sum_{x=0}^{N-1} \xi_t(x) \delta_{x/N}, \quad (12)$$

where $\delta_{x/N}$ is the Dirac measure at x/N , converges in probability, on $D([0, T], M_+(\mathbb{T}) \times M_+(\mathbb{T}))$ (the space of right continuous functions with left limits taking values in $M_+(\mathbb{T}) \times M_+(\mathbb{T})$ with $M_+(\mathbb{T})$ is the space of finite positive measures on the torus $\mathbb{T} = [0, 1)$ endowed with the weak topology), as N goes to infinity, to a deterministic measure, absolutely continuous with respect to the Lebesgue measure, $(\rho_1(t, u) du, \rho_2(t, u) du)$, with density $(\rho_1(\cdot, \cdot), \rho_2(\cdot, \cdot))$ solution of the reaction-diffusion system (16). The strategy consists in studying the entropy of the process with respect to Poisson measures with parameter the expected “good profile” $(\rho_1(\cdot, \cdot), \rho_2(\cdot, \cdot))$.

For a density profile $\rho_1(\cdot) \times \rho_2(\cdot)$, on $\mathbb{T} \times \mathbb{T}$, we denote by $\nu_{\rho_1(\cdot)}^N \times \nu_{\rho_2(\cdot)}^N$ the product of Poisson measures such that, for all $x \in \mathbb{T}_N$ and $k, j \in \mathbb{N}$,

$$\begin{aligned} \left(\nu_{\rho_1(\cdot)}^N \times \nu_{\rho_2(\cdot)}^N \right) \{(\eta, \xi), \eta(x) = k, \xi(x) = j\} &= \frac{(\rho_1(x/N))^k}{k!} \exp(-\rho_1(x/N)) \\ &\times \frac{(\rho_2(x/N))^j}{j!} \exp(-\rho_2(x/N)). \end{aligned}$$

The family of measures $(\nu_{\rho}^N \times \nu_{\rho}^N)$ with constant parameter $\rho > 0$ is invariant for the independent random walks which govern the migration of individuals, this is why we study the entropy variation with respect to these reference measures.

We define the entropy of μ^N on $\mathbb{N}^{\mathbb{T}_N} \times \mathbb{N}^{\mathbb{T}_N}$ with respect to $(\rho^1(\cdot), \rho^2(\cdot))$ by

$$H \left[\mu^N | \nu_{\rho_1(\cdot)}^N \times \nu_{\rho_2(\cdot)}^N \right] = \int \log \left(\frac{d\mu^N}{d(\nu_{\rho_1(\cdot)}^N \times \nu_{\rho_2(\cdot)}^N)} \right) d\mu^N(\eta, \xi). \quad (13)$$

For a cylinder function h on $\mathbb{N}^{\mathbb{T}_N} \times \mathbb{N}^{\mathbb{T}_N}$

$$\tilde{h}(a, b) = \int h(\eta, \xi) d(\nu_a^N \times \nu_b^N)(\eta, \xi). \quad (14)$$

Theorem 3 Assume that there exists smooth positive functions $m_1(\cdot)$ and $m_2(\cdot)$, defined on the torus \mathbb{T} , such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} H \left[\mu^N | \nu_{m_1(\cdot)}^N \times \nu_{m_2(\cdot)}^N \right] = 0. \quad (15)$$

Then for all functions $G_1(\cdot)$ and $G_2(\cdot)$ continuous on \mathbb{T} , $\delta > 0$ and $t \in [0, T]$, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \mu^N S_t^N \left\{ (\eta, \xi) : \left| \frac{1}{N} \sum_{x=0}^{N-1} \eta(x) G_1(x/N) - \int_0^1 G_1(\theta) \lambda_1(t, \theta) d\theta \right| > \delta \right. \\ \left. \text{and } \left| \frac{1}{N} \sum_{x=0}^{N-1} \xi(x) G_2(x/N) - \int_0^1 G_2(\theta) \lambda_2(t, \theta) d\theta \right| > \delta \right\} = 0 \end{aligned}$$

where $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ is the unique smooth solution of the system:

$$\partial_t \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{2} \Delta \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} \tilde{\beta}_1(\lambda_1, \lambda_2) - \tilde{\delta}_1(\lambda_1, \lambda_2) + \tilde{g}(\lambda_1, \lambda_2) \\ \tilde{\beta}_2(\lambda_1, \lambda_2) - \tilde{\delta}_2(\lambda_1, \lambda_2) - \tilde{g}(\lambda_1, \lambda_2) \end{pmatrix}, \quad (16)$$

with initial conditions $\lambda_1(0, d\theta) = m_1(\theta)$, and $\lambda_2(0, d\theta) = m_2(\theta)$; and g is a function on $\mathbb{N} \times \mathbb{N}$ defined by

$$g(\eta(z), \xi(z)) = \left(1 - \phi \mathbf{1}_{\{\eta(z) > 0\}}\right) \xi(z) - \mathbf{1}_{\{\eta(z) > 0\}} \left(\lambda \mathbf{1}_{\{\xi(z)=0\}} + \beta \mathbf{1}_{\{\xi(z) > 0\}}\right) \sum_{y \sim z} \xi(y). \quad (17)$$

Extension to infinite volume

By a coupling method, we will extend Theorem 3 to infinite volume. We will prove that two processes, one defined on \mathbb{Z} and the other one on $\mathbb{T}_{CN} = \{-CN, \dots, CN\}$, are “close” when C is large. Following Landim and Yau (1995) we define the specific entropy of a measure μ with respect to a measure ν on \mathbb{Z}

$$\mathcal{H}_N[\mu | \nu] = \frac{1}{N} \sum_{n \geq 1} H[\mu^n | \nu^n] e^{-\theta n/N}, \quad (18)$$

where $\theta > 0$ is fixed and μ^n and ν^n are the respective restrictions of μ and ν to $\Lambda_n = \{-n, \dots, n\}$. Let \tilde{S}_t^N be the semi-group associated to the generator Ω of the process $(\eta_t, \xi_t)_{t \geq 0}$ given in (7).

Theorem 4 *We consider a sequence of initial distributions $(\mu^N)_{N \in \mathbb{N}}$ on $\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}}$ such that there exists $M > 0$ with $\mu^N(\eta(x) + \xi(x)) \leq M$ for all $x \in \mathbb{Z}$, and smooth positive functions $m_1(\cdot)$ and $m_2(\cdot)$, defined on \mathbb{R} , satisfying*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \mathcal{H}_N \left[\mu^N | \nu_{m_1(\cdot)}^N \times \nu_{m_2(\cdot)}^N \right] = 0. \quad (19)$$

Then for all functions $G_1(\cdot)$ and $G_2(\cdot)$ continuous on \mathbb{R} , $\delta > 0$ and $t \in [0, T]$, we have

$$\lim_{N \rightarrow \infty} \mu^N \tilde{S}_t^N \left\{ (\eta, \xi) : \left| \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta(x) G_1(x/N) - \int_{\mathbb{R}} G_1(\theta) \lambda_1(t, \theta) d\theta \right| > \delta \right. \\ \left. \text{and} \quad \left| \frac{1}{N} \sum_{x \in \mathbb{Z}} \xi(x) G_2(x/N) - \int_{\mathbb{R}} G_2(\theta) \lambda_2(t, \theta) d\theta \right| > \delta \right\} = 0$$

where $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ is the unique smooth solution of the system (16), with initial conditions $\lambda_1(0, d\theta) = m_1(\theta)$, and $\lambda_2(0, d\theta) = m_2(\theta)$.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. The aim of this section is to prove that, when $\phi + 2d\lambda < 1$ the processes converge to the all “0” configuration, this result will be deduce from an ergodicity criterion established in Chen (1992).

For any integer $n \geq 1$, we set $\Lambda_n = \{-n, \dots, n\}^d$, $\tilde{\lambda} = 2d\lambda$, $\tilde{\beta} = 2d\beta$. We consider the sequence of processes $(\xi_t^n)_{n \geq 0}$ (respectively, $(\eta_t^n)_{n \geq 0}$) defined on \mathbb{N}^{Λ_n} as the restriction of ξ_t (respectively, η_t) to Λ_n with generator $\hat{\Omega}_n$ (respectively, $\tilde{\Omega}_n$). For any cylinder function f of the configuration ξ ,

$$\hat{\Omega}_n f(\xi) = \hat{\Omega}_n^1 f(\xi) + \Omega_n^2 f(\xi),$$

$$\hat{\Omega}_n^1 f(\xi) = \phi \sum_{x \in \Lambda_n} \xi(x) \left[f(\xi^{x,+}) - f(\xi) \right] + \sum_{x \in \Lambda_n} \xi(x) \left[f(\xi^{x,-}) - f(\xi) \right]$$

and

$$\Omega_n^2 f(\xi) = \sum_{x \in \Lambda_n} \left(\sum_{\substack{y \sim x \\ y \in \Lambda_n}} \xi(y) \right) \left(\lambda \mathbf{1}_{\{\xi(x)=0\}} + \beta \mathbf{1}_{\{\xi(x) > 0\}} \right) \left[f(\xi^{x,+}) - f(\xi) \right] \\ = \sum_{x, y \in \Lambda_n} p(y, x) \xi(y) \left(\tilde{\lambda} \mathbf{1}_{\{\xi(x)=0\}} + \tilde{\beta} \mathbf{1}_{\{\xi(x) > 0\}} \right) \left[f(\xi^{x,+}) - f(\xi) \right].$$

For any cylinder function f of the configuration η ,

$$\bar{\Omega}_n f(\eta) = \bar{\Omega}_n^1 f(\eta) + \Omega_n^2 f(\eta),$$

where

$$\bar{\Omega}_n^1 f(\eta) = \phi \sum_{x \in \Lambda_n} \eta(x) [f(\eta^{x,+}) - f(\eta)] + \sum_{x \in \Lambda_n} [f(\eta^x) - f(\eta)].$$

Given a constant $M > 1$ which can be as close to 1 as desired, we set

$$k_x = \sum_{n=0}^{\infty} M^{-n} p^{(n)}(x, 0) \quad \text{for all } x \in \mathbb{Z}. \quad (20)$$

Since $p(x, y)$ is translation invariant with $p(x, x) = 0$, we have

$$\sum_{y \in \mathbb{Z}} p(x, y) k_y \leq M k_x \quad \text{and} \quad \sum_{x \in \mathbb{Z}} k_x < +\infty, \quad (21)$$

Now, given a site $x \in \mathbb{Z}$ and two configurations ξ_1 and ξ_2 , we set

$$\rho_x(\xi_1(x)) = \xi_1(x), \quad q_x(\xi_1) = \xi_1(x) k_x \quad \text{and} \quad q_x(\xi_1, \xi_2) = |\xi_1(x) - \xi_2(x)| k_x.$$

We construct our processes on

$$E_0 = \{\xi \in E = \mathbb{N}^{\mathbb{Z}} : q(\xi) = \sum_{x \in \mathbb{Z}} \rho_x(\xi(x)) k_x < \infty\}.$$

The following theorem is an adaptation of Theorems 13.8, 13.18 and 14.3.

Theorem 5 *For every $1 \leq n \leq m$, there exists a coupling generator $\hat{\Omega}_{n,m}$ of $\hat{\Omega}_n$ and $\hat{\Omega}_m$ such that for any $z \in \Lambda_n$ and any $\xi_1, \xi_2 \in E_0$,*

$$\hat{\Omega}_{n,m} q_z(\xi_1, \xi_2) \leq \sum_{x \in \Lambda_n} c_{xz} q_x(\xi_1, \xi_2) + \sum_{x \in \Lambda_m \setminus \Lambda_n} g_{xz} q_x(\xi_2), \quad (22)$$

where the non-diagonal elements of matrices $(c_{xy})_{x,y \in \Lambda_n}$, $(g_{xy})_{x,y \in \Lambda_m}$, are all non-negative, and

$$\lim_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (c_{xy} + g_{xy}) < +\infty. \quad (23)$$

Assume additionally that the coefficients (c_{xy}) given in (22) also satisfy

$$\exists \alpha > 0, \quad \lim_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \sum_{y \in \Lambda_n} c_{xy} < -\alpha < 0, \quad (24)$$

$$\exists K < \infty, \quad \lim_{n \rightarrow \infty} \sup_{x \in \Lambda_n} \sum_{y \in \Lambda_n} |c_{xy}| < K. \quad (25)$$

Then the Markov process $(\xi_t)_{t \geq 0}$ has at most one stationary distribution π on (E, \mathcal{E}) satisfying

$$\pi q = \int_{E_0} \pi(d\zeta) q(\zeta) < \infty. \quad (26)$$

Our main tool is to use repeatedly basic coupling of the different generators describing all the aspects of the model under study. We write in detail the first one, the others are built in the same spirit.

To check Condition (22), we use basic coupling. Let n and m be two integers such that $1 \leq n \leq m$. We denote by $\hat{\Omega}_{n,m}^1$ the coupled generator associated to $\hat{\Omega}_n^1$ and $\hat{\Omega}_m^1$, and by $\Omega_{n,m}^2$ the coupled

generator associated to Ω_n^2 and Ω_m^2 . In the same way, we define the coupled generator $\bar{\Omega}_{n,m}^1$. To lighten our calculations, we set

$$a_i(x, y) = \xi_i(y) \left(\lambda \mathbf{1}_{\{\xi_i(x)=0\}} + \beta \mathbf{1}_{\{\xi_i(x)>0\}} \right) \quad \text{for } i = 1, 2 \text{ and } x, y \in \mathbb{Z}.$$

We define the coupling $\Omega_{n,m}^2$ describing the infections originated from neighboring sites as follows

$$\begin{aligned} \Omega_{n,m}^2 f(\xi_1, \xi_2) &= 2d \sum_{x,y \in \Lambda_n} p(x, y) (a_1(x, y) \wedge a_2(x, y)) \left[f(\xi_1^{y,+}, \xi_2^{y,+}) - f(\xi_1, \xi_2) \right] \\ &+ 2d \sum_{x,y \in \Lambda_n} p(x, y) (a_1(x, y) - a_2(x, y))^+ \left[f(\xi_1^{y,+}, \xi_2) - f(\xi_1, \xi_2) \right] \\ &+ 2d \sum_{x,y \in \Lambda_n} p(x, y) (a_2(x, y) - a_1(x, y))^+ \left[f(\xi_1, \xi_2^{y,+}) - f(\xi_1, \xi_2) \right] \\ &+ 2d \sum_{x \in \Lambda_m \setminus \Lambda_n} \sum_{y \in \Lambda_m} p(x, y) a_2(x, y) \left[f(\xi_1, \xi_2^{y,+}) - f(\xi_1, \xi_2) \right] \\ &+ 2d \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_m \setminus \Lambda_n} p(x, y) a_2(x, y) \left[f(\xi_1, \xi_2^{y,+}) - f(\xi_1, \xi_2) \right] \end{aligned}$$

The coupled generator $\Omega_{n,m}^2$ describes the outside infections from site x to y where $x \sim y$ and $(x, y \in \Lambda_n)$ or $(x \in \Lambda_m \setminus \Lambda_n \text{ and } y \in \Lambda_m)$ or $(x \in \Lambda_n \text{ and } y \in \Lambda_m \setminus \Lambda_n)$, for the processes whose generators are Ω_n^2 and Ω_m^2 .

We now deal with the coupled generators $\hat{\Omega}_{n,m}^1$ (respectively, $\bar{\Omega}_{m,n}^1$) defined as the sum of $\hat{\Omega}_{n,m}^{1,i}$, $i = 1, 2$ (respectively, $\bar{\Omega}_{n,m}^{1,i}$, $i = 1, 2$) with the coupled generator $\bar{\Omega}_{n,m}^{1,1}$ (respectively, $\hat{\Omega}_{n,m}^{1,2}$) describing the within infections (respectively, recoveries) at site $x \in \Lambda_n$. However, the following coupled generators

$$\bar{\Omega}_{n,m}^{1,1} f(\eta_1, \eta_2) = \hat{\Omega}_{n,m}^{1,1} f(\eta_1, \eta_2) \quad \text{and} \quad \bar{\Omega}_{n,m}^{1,2} f(\eta_1, \eta_2) = \sum_{x \in \Lambda_m} \left[f(\eta_1^x, \eta_2^x) - f(\eta_1, \eta_2) \right],$$

describe the within infections and cluster recovery in a given site for the CRP(∞). For sites $x, y, z \in \mathbb{Z}$, we set

$$\begin{aligned} b_z(x, y) &= (a_1(x, y) - a_2(x, y))^+ [q_z(\xi_1^{x,+}, \xi_2) - q_z(\xi_1, \xi_2)] \\ &+ (a_2(x, y) - a_1(x, y))^+ [q_z(\xi_1, \xi_2^{x,+}) - q_z(\xi_1, \xi_2)]. \end{aligned}$$

First of all, we observe that

$$q_z(\xi_1^{x,+}, \xi_2) = q_z(\xi_1, \xi_2^{x,-}) = \begin{cases} q_z(\xi_1, \xi_2) + k_z & \text{when } \xi_1(x) \geq \xi_2(x) \text{ and } x = z \\ q_z(\xi_1, \xi_2) - k_z & \text{when } \xi_1(x) < \xi_2(x) \text{ and } x = z \\ 0 & \text{when } x \neq z, \end{cases} \quad (27)$$

while

$$q_z(\xi_1, \xi_2^{x,+}) = q_z(\xi_1^{x,-}, \xi_2) = \begin{cases} q_z(\xi_1, \xi_2) + k_z & \text{when } \xi_1(x) \leq \xi_2(x) \text{ and } x = z \\ q_z(\xi_1, \xi_2) - k_z & \text{when } \xi_1(x) > \xi_2(x) \text{ and } x = z \\ 0 & \text{when } x \neq z. \end{cases} \quad (28)$$

In particular, by decomposing according to whether $\xi_1(x)$ and $\xi_2(x)$ are different from or equal to 0, we obtain

$$\begin{aligned} b_z(x, y) &= \lambda |\xi_1(y) - \xi_2(y)| k_z \mathbf{1}_{\{\xi_1(x)=\xi_2(x)=0\}} + \beta |\xi_1(y) - \xi_2(y)| k_z \mathbf{1}_{\{\xi_1(x)>0, \xi_2(x)>0\}} \\ &+ (\beta \xi_2(y) - \lambda \xi_2(y)) k_z \mathbf{1}_{\{\xi_2(x)>\xi_1(x)=0\}} + (\beta \xi_1(y) - \lambda \xi_1(y)) k_z \mathbf{1}_{\{\xi_1(x)>\xi_2(x)=0\}} \\ &\leq \lambda |\xi_1(y) - \xi_2(y)| k_z = q_y(\xi_1, \xi_2) k_z / k_y \end{aligned}$$

when $x = z$, and $b_z(x, y) = 0$ when $x \neq z$. We conclude that

$$b_z(x, y) \leq \lambda q_y(\xi_1, \xi_2) k_z/k_y \quad \text{if } x = z \quad \text{and} \quad b_z(x, y) = 0 \quad \text{if } x \neq z. \quad (29)$$

By using (29), we obtain that for any site $z \in \Lambda_n \subset \Lambda_m$,

$$\begin{aligned} \Omega_{n,m}^2 q_z(\xi_1, \xi_2) &= 2d \sum_{y \in \Lambda_n} p(z, y) b_z(z, y) \\ &\quad + 2d \sum_{y \in \Lambda_m \setminus \Lambda_n} p(z, y) a_2(z, y) [q_z(\xi_1, \xi_2^{z,+}) - q_z(\xi_1, \xi_2)] \\ &\leq 2d \lambda \sum_{y \in \Lambda_n} p(z, y) q_y(\xi_1, \xi_2) k_z/k_y \\ &\quad + 2d \lambda \sum_{y \in \Lambda_m \setminus \Lambda_n} p(z, y) q_y(\xi_2) k_z/k_y. \end{aligned}$$

Now, assume that $\xi_1(z) > \xi_2(z)$ for some $z \in \Lambda_n$. From (27) and (28), it follows that

$$\begin{aligned} (\hat{\Omega}_{n,m}^{1,1} + \hat{\Omega}_{n,m}^{1,2}) q_z(\xi_1, \xi_2) &= \phi (\xi_1(z) - \xi_2(z)) k_z \\ &\quad - (\xi_1(z) - \xi_2(z)) k_z = (\phi - 1) q_z(\xi_1, \xi_2) \end{aligned}$$

The same holds when $\xi_1(z) < \xi_2(z)$. In particular,

$$(\hat{\Omega}_{n,m}^{1,1} + \hat{\Omega}_{n,m}^{1,2}) q_z(\xi_1, \xi_2) \leq (\phi - 1) q_z(\xi_1, \xi_2)$$

in any case since both members of the inequality are equal to 0 when $\xi_1(z) = \xi_2(z)$. Finally, by observing that

$$q_z(\xi_1^x, \xi_2^x) = \begin{cases} q_z(\xi_1, \xi_2) & \text{when } x \neq z \\ 0 & \text{when } x = z, \end{cases}$$

we have

$$\bar{\Omega}_{n,m}^{1,2} q_z(\eta_1, \eta_2) = -q_z(\eta_1, \eta_2).$$

Putting things together, we get the upper bound

$$\hat{\Omega}_{n,m} q_z(\xi_1, \xi_2) \leq \sum_{y \in \Lambda_n} c_{yz} q_y(\xi_1, \xi_2) + \sum_{y \in \Lambda_m \setminus \Lambda_n} g_{yz} q_y(\xi_2) \quad (30)$$

where the coefficients c_{yz} and g_{yz} are given by

$$c_{yz} = \begin{cases} \phi - 1 & \text{if } y = z \\ \tilde{\lambda} p(z, y) k_z/k_y & \text{if } y \neq z \end{cases} \quad \text{and} \quad g_{yz} = \tilde{\lambda} p(z, y) k_z/k_y.$$

Inequality (30) also holds for the coupled generator $\bar{\Omega}_{n,m}$. Condition (22) of Theorem 5 is then satisfied. By (20) and (21), for any site $y \in \Lambda_m$ and any constant $M > 1$, we have

$$\sum_{z \in \Lambda_n} c_{yz} \leq \phi - 1 + 2d \lambda \sum_{z \in \Lambda_n} p(z, y) \frac{k_z}{k_y} \leq \phi - 1 + 2d \lambda M.$$

In particular, condition (24) in Theorem 5 holds whenever $\phi + 2d \lambda < 1$. In other respects conditions (23) and (25) are trivial. This completes the proof of Theorem 1.

Proof of Theorem 2. We start by proving the first statement, i.e., if $\lambda > \lambda_c$ an epidemic may occur. First, we note that by using basic coupling, the probability of an epidemic is nondecreasing with respect to the initial configuration and to each of the parameters λ , β , and ϕ .

Lemma 3.1 *The $IRP(\infty)$ and $CRP(\infty)$ are attractive and monotone with respect to the parameters λ , β and ϕ .*

Again by using basic coupling of the basic contact process with parameter λ and the $IRP(\infty)$ (respectively, $CRP(\infty)$) with parameter (λ, β, ϕ) we show that both $IRP(\infty)$ and $CRP(\infty)$ have more infected individuals than the contact process. This together, with Lemma 3.1, implies that, when $\lambda > \lambda_c$, an epidemic may occur for any $\phi \geq 0$ and $\beta \geq 0$ for both processes. To prove the second statement, we will show that there exist $\phi_1 > 0$ and $\phi_2 < \infty$ such that if $\phi < \phi_1$ there is no epidemic, while if $\phi > \phi_2$ an epidemic may occur. Due to the monotonicity with respect to the within infection rate ϕ , this will imply the existence of $\phi_c \in [\phi_1, \phi_2]$ such that Theorem 2 holds.

The existence of ϕ_1 follows from the fact that when $\phi + 2d(\lambda \vee \beta) < 1$, the $CRP(\infty)$ and $IRP(\infty)$ converge to the “all 0” configuration. In others words, if (5) holds then there is not epidemic for the $IRP(\infty)$ and the $CRP(\infty)$. We now deal with the existence of ϕ_2 . Let ξ_t^1 denote the $IRP(\infty)$ with parameters $(\lambda, 0, \phi)$ and ξ_t^2 denote the $IRP(\kappa)$ with parameters $(\lambda, 0, \phi)$. Using basic coupling we prove that if $\xi_0^1(x) \geq \xi_0^2(x)$ for any $x \in \mathbb{Z}$ at time 0, then ξ_t^1 and ξ_t^2 can be constructed on the same probability space in such way that

$$P_{(\xi_0^1, \xi_0^2)}(\forall x \in \mathbb{Z}, \xi_t^1(x) \geq \xi_t^2(x)) = 1,$$

where $P_{(\xi_0^1, \xi_0^2)}$ is the law of the coupled process starting from (ξ_0^1, ξ_0^2) . It follows that the $IRP(\infty)$ with parameters $(\lambda_1, 0, \phi_1)$ has more infected individuals than the $IRP(\kappa)$ with parameters $(\lambda_2, 0, \phi_2)$. It follows that by Theorem 3, Belhadji and Lanchier (2006), an epidemic may occur for the $IRP(\kappa)$ with parameters (λ, β, ϕ) for all $\kappa > 1$, provided the within infection rate ϕ is greater than some critical value. By Lemma 3.1 the existence of ϕ_2 such that an epidemic may occur for the $IRP(\infty)$ with parameters (λ, β, ϕ) , for all $\phi \geq \phi_2$ and $\beta \leq \lambda$ follows. In the same way, that is by basic coupling we obtain that the $CRP(\infty)$ with parameters (λ, β, ϕ) has more infected individuals than the $CRP(\kappa)$ with parameters $(\lambda, 0, \phi)$. Due to monotonicity of the $CRP(\kappa)$ with respect to κ , we can fix κ such that $\kappa\lambda > \lambda_c$, and apply Theorem 1, Schinazi (2002) and Lemma 3.1, to get the existence of ϕ_2 such that the $CRP(\infty)$ with parameters (λ, β, ϕ) is not ergodic for all $\phi \geq \phi_2$ and $\beta \geq 0$. Thus Theorem 2 follows.

4. Proof of theorems 3 and 4

4.1. One block estimate. It allows the replacement of a local function $h(\eta(x), \xi(x))$, $x \in \mathbb{Z}$ by a function of the *empirical density* $\eta^k(x)$ (respectively, $\xi^k(x)$) of healthy (respectively, infected) individuals in a box of length $2k + 1$, $k \in \mathbb{N}$ centered at x :

$$\eta^k(x) = \frac{1}{2k+1} \sum_{|y-x| \leq k} \eta(y) \quad \text{and} \quad \xi^k(x) = \frac{1}{2k+1} \sum_{|y-x| \leq k} \xi(y). \quad (31)$$

Proposition 4.1 [One block estimate](Mourragui, 1996) *Let h be a bounded function on $\mathbb{N} \times \mathbb{N}$. We have*

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \int_{\mathbb{N}^{\mathbb{T}_N}} \left(\frac{1}{N} \sum_{x=0}^{N-1} \int_0^T V_k(\eta_s(x), \xi_s(x)) ds \right) d\mu^N = 0,$$

where

$$V_k(\eta(x), \xi(x)) = \left| \frac{1}{2k+1} \sum_{|x-y| \leq k} (h(\eta(y), \xi(y)) - \tilde{h}(\eta^k(x), \xi^k(x))) \right|.$$

We refer to Mourragui (1996) for its proof.

In what follows, we will use intensively the following change of variables formulas stated for each cylinder function on $\mathbb{N}^{\mathbb{T}^N} \times \mathbb{N}^{\mathbb{T}^N}$. Let $\nu_a^N \times \nu_b^N$ be the product of Poisson measures on $\mathbb{N}^{\mathbb{T}^N} \times \mathbb{N}^{\mathbb{T}^N}$, with arbitrary parameters $a > 0$ and $b > 0$ respectively. We have:

$$\begin{aligned} \int f(\eta^{x,-}, \xi^{x,+}) d(\nu_a^N \times \nu_b^N)(\eta, \xi) &= \frac{a}{b} \int \frac{\xi(x)}{1 + \eta(x)} f(\eta, \xi) d(\nu_a^N \times \nu_b^N)(\eta, \xi), \\ \int f(\eta^{x,+}, \xi^{x,-}) d(\nu_a^N \times \nu_b^N)(\eta, \xi) &= \frac{b}{a} \int \frac{\eta(x)}{1 + \xi(x)} f(\eta, \xi) d(\nu_a^N \times \nu_b^N)(\eta, \xi). \end{aligned} \quad (32)$$

We will use the entropy inequality: If α^N and β^N are two measures on $\mathbb{N}^{\mathbb{T}^N} \times \mathbb{N}^{\mathbb{T}^N}$, then for all bounded function U and $\alpha > 0$,

$$\int U d\beta^N \leq \frac{1}{\alpha} \log \int \exp(\alpha U) d\alpha^N + \frac{1}{\alpha} H[\beta^N | \alpha^N]. \quad (33)$$

4.2. Proof of Theorem 3 It is divided in several lemmas. The objective is to prove that

$$\lim_{N \rightarrow +\infty} \mu_t^N(A_N^{G_1, G_2, \delta}) = 0, \quad (34)$$

where

$$\begin{aligned} A_N^{G_1, G_2, \delta} &= \left\{ (\eta, \xi) : \left| \frac{1}{N} \sum_{x=0}^{N-1} \eta(x) G_1(x/N) - \int_0^1 G_1(\theta) \lambda_1(t, \theta) d\theta \right| > \delta \right. \\ &\quad \left. \text{and} \quad \left| \frac{1}{N} \sum_{x=0}^{N-1} \xi(x) G_2(x/N) - \int_0^1 G_2(\theta) \lambda_2(t, \theta) d\theta \right| > \delta \right\}, \end{aligned} \quad (35)$$

and $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ is the solution of (16). The entropy inequality (33) allows us to write

$$\mu_t^N(A_N^{G_1, G_2, \delta}) \leq \frac{\frac{1}{N} \log 2 + \frac{1}{N} H[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N]}{\frac{1}{N} \log [1 + \{\nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N(A_N^{G_1, G_2, \delta})\}^{-1}]}. \quad (36)$$

Proof of (34) relies on the following steps.

Proposition 4.2 For each t in $[0, T]$, there exists a function A_N^t which converges to zero when N goes to infinity and a constant C such that

$$\frac{1}{N} H[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N] \leq A_N^t + \frac{C}{N} \int_0^t H[\mu_s^N | \nu_{\lambda_1(s, \cdot)}^N \times \nu_{\lambda_2(s, \cdot)}^N] ds. \quad (37)$$

Using Varadhan theorem (Chen, 1992 page 286), we have for all profiles $\rho_1(\cdot)$ and $\rho_2(\cdot)$ and $\delta > 0$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log (\nu_{\rho_1(t, \cdot)}^N \times \nu_{\rho_2(t, \cdot)}^N)(A_N^{G_1, G_2, \delta}) < 0. \quad (38)$$

By using (37) and applying Gronwall lemma we then prove that:

$$\lim_{N \rightarrow +\infty} \frac{1}{N} H[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N] = 0. \quad (39)$$

Inequality (36), (39) and (38) imply (34).

In the proof of the proposition 4.2, we will need to write $\lambda_1(t, \cdot)^{-1}$ and $\lambda_2(t, \cdot)^{-2}$, and to avoid technical difficulties, we assume that $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ the solution of (16) is bounded below by a strictly positive constant K :

$$\inf_{t \geq 0} \inf_{x \in \mathbb{T}_N} \lambda_i(t, x/N) = K, \quad \text{for } i = 1, 2.$$

Indeed, if it is not the case, the proof may be modified by replacing $\lambda_1(t, \cdot)$ and $\lambda_2(t, \cdot)$ by $\lambda_1(t, \cdot) + \varepsilon$ and $\lambda_2(t, \cdot) + \varepsilon$ ($\varepsilon > 0$) and by letting ε goes to zero. In order to prove the proposition we need also to compute the relative entropy $H \left[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N \right]$.

We denote by f^N and f_t^N the Radon-Nikodym derivatives of μ^N and $\mu_t^N = \mu^N S_t^N$ with respect to the reference measure $(\nu_\rho^N \times \nu_\rho^N)$. Let ψ_t^N denote the Radon-Nikodym derivative of $\nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N$ with respect to the reference measure. Because $\nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N$ and $\nu_\rho^N \times \nu_\rho^N$ are product measures, ψ_t^N can be computed explicitly

$$\begin{aligned} \psi_t^N(\eta, \xi) = & \exp \left(\sum_{i=0}^{N-1} \left\{ \eta(i) \log \left(\frac{\lambda_1(t, i/N)}{\rho} \right) + \rho - \lambda_1(t, i/N) \right\} \right) \\ & \times \exp \left(\sum_{i=0}^{N-1} \left\{ \xi(i) \log \left(\frac{\lambda_2(t, i/N)}{\rho} \right) + \rho - \lambda_2(t, i/N) \right\} \right). \end{aligned} \quad (40)$$

PROOF. (Proposition 4.2) We derivate the relative entropy, using that the density f_t^N is the solution of the Kolmogorov forward equation $\partial_t f_t^N = \Omega_N^* f_t^N$.

$$\begin{aligned} \frac{d}{dt} H \left[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N \right] &= \frac{d}{dt} \int f_t^N \log \left(\frac{f_t^N}{\psi_t^N} \right) d(\nu_\rho^N \times \nu_\rho^N) \\ &= \int f_t^N \Omega_N \log \left(\frac{f_t^N}{\psi_t^N} \right) d(\nu_\rho^N \times \nu_\rho^N) - \int \frac{f_t^N}{\psi_t^N} \frac{d}{dt} (\psi_t^N) d(\nu_\rho^N \times \nu_\rho^N) \\ &= N^2 \int f_t^N \Omega_N^{\mathcal{D},1} \log \left(\frac{f_t^N}{\psi_t^N} \right) d(\nu_\rho^N \times \nu_\rho^N) \\ &\quad + N^2 \int f_t^N \Omega_N^{\mathcal{D},2} \log \left(\frac{f_t^N}{\psi_t^N} \right) d(\nu_\rho^N \times \nu_\rho^N) \\ &\quad + \int f_t^N \Omega_N^{\mathcal{R}} \log \left(\frac{f_t^N}{\psi_t^N} \right) d(\nu_\rho^N \times \nu_\rho^N) - \int \frac{f_t^N}{\psi_t^N} \frac{d}{dt} (\psi_t^N) d(\nu_\rho^N \times \nu_\rho^N) \\ &= I_1 + I_2 + I_3 - I_4. \end{aligned} \quad (41)$$

To compute I_1 and I_2 we use the explicit expression for ψ_t^N given in (40), the fact that $\Omega_N^{\mathcal{D},1}$ and $\Omega_N^{\mathcal{D},2}$ are self-adjoint with respect to the product measure $\nu_\rho^N \times \nu_\rho^N$, and

$$x [\log(y) - \log(x)] \leq y - x, \quad \text{for all } x, y > 0, \quad (42)$$

$$\sum_{i=0}^{N-1} [\lambda_1(t, (i+1)/N) + \lambda_1(t, (i-1)/N) - 2\lambda_1(t, i/N)] = 0. \quad (43)$$

We obtain that

$$\begin{aligned} I_1 &\leq N^2 \int \frac{f_t^N}{\psi_t^N} \Omega_N^{\mathcal{D},1} (\psi_t^N) d(\nu_\rho^N \times \nu_\rho^N) \\ &\leq \frac{N^2}{2} \sum_{|j-i|=1} \int \eta(i) \left(\frac{\lambda_1(t, j/N)}{\lambda_1(t, i/N)} - 1 \right) d\mu_t^N(\eta, \xi) \\ &= \frac{N^2}{2} \sum_{i=0}^{N-1} \int \left(\frac{\eta(i)}{\lambda_1(t, i/N)} - 1 \right) [\lambda_1(t, (i+1)/N) + \lambda_1(t, (i-1)/N) - 2\lambda_1(t, i/N)] d\mu_t^N(\eta, \xi) \end{aligned}$$

To take advantage of the fact that $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ is solution of (16) and to conjure up the Laplacian of $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ that will appear later in I_4 with negative sign, we observe that a Taylor-Young expansion gives:

$$N^2 \left[\lambda_1(t, (i+1)/N) + \lambda_1(t, (i-1)/N) - 2\lambda_1(t, i/N) \right] = \frac{\partial^2}{\partial \theta^2} \lambda_1(t, i/N) + o(1/N^2).$$

So,

$$I_1 \leq \frac{1}{2} \sum_{i=0}^{N-1} \int \left(\frac{\eta(i)}{\lambda_1(t, i/N)} - 1 \right) \frac{\partial^2}{\partial \theta^2} \lambda_1(t, i/N) d\mu_t^N(\eta, \xi) + o(1/N).$$

The second term I_2 has a similar upper bound:

$$I_2 \leq \frac{1}{2} \sum_{i=0}^{N-1} \int \left(\frac{\xi(i)}{\lambda_2(t, i/N)} - 1 \right) \frac{\partial^2}{\partial \theta^2} \lambda_2(t, i/N) d\mu_t^N(\eta, \xi) + o(1/N).$$

To deal with the third term I_3 , we apply inequality (42), and by using substitution rule (32) we get:

$$\begin{aligned} I_3 &\leq \sum_{i=0}^{N-1} \int \left[\frac{\eta(i)}{\lambda_1(t, i/N)} \beta_1(\eta(i) - 1, \xi(i)) - \beta_1(\eta(i), \xi(i)) \right] d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left[\frac{\lambda_1(t, i/N)}{\eta(i) + 1} \delta_1(\eta(i) + 1, \xi(i)) - \delta_1(\eta(i), \xi(i)) \right] d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left[\frac{\xi(i)}{\lambda_2(t, i/N)} \beta_2(\eta(i), \xi(i) - 1) - \beta_2(\eta(i), \xi(i)) \right] d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left[\frac{\lambda_1(t, i/N)}{\xi(i) + 1} \delta_2(\eta(i), \xi(i) + 1) - \delta_2(\eta(i), \xi(i)) \right] d\mu_t^N(\eta, \xi) \\ &+ \phi \sum_{i=0}^{N-1} \int \left[\frac{\lambda_1(t, i/N)}{\lambda_2(t, i/N)} \times \frac{\xi(i)(\xi(i) - 1)}{\eta(i) + 1} - \xi(i) \mathbf{1}_{\{\eta(i) > 0\}} \right] d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left(\frac{\lambda_1(t, i/N)}{\lambda_2(t, i/N)} \times \frac{\xi(i)}{\eta(i) + 1} \mathbf{1}_{\{\xi(i)=1\}} - \mathbf{1}_{\{\eta(i) > 0, \xi(i)=0\}} \right) \times \left(\lambda \sum_{j \sim i} \xi(j) \right) d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left(\frac{\lambda_1(t, i/N)}{\lambda_2(t, i/N)} \times \frac{\xi(i)}{\eta(i) + 1} - \mathbf{1}_{\{\eta(i) > 0, \xi(i) > 0\}} \right) \times \left(\beta \sum_{j \sim i} \xi(j) \right) d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left[\frac{\lambda_2(t, i/N)}{\lambda_1(t, i/N)} \eta(i) - \xi(i) \right] d\mu_t^N(\eta, \xi). \end{aligned}$$

We rewrite the fourth term I_4 using that $(\lambda_1(t, \cdot), \lambda_2(t, \cdot))$ solves equation (16)

$$\begin{aligned} I_4 &= \sum_{i=0}^{N-1} \int \left[\left(\frac{\eta(i)}{\lambda_1(t, i/N)} - 1 \right) \frac{d}{dt} \lambda_1(t, i/N) + \left(\frac{\xi(i)}{\lambda_2(t, i/N)} - 1 \right) \frac{d}{dt} \lambda_2(t, i/N) \right] d\mu_t^N(\eta, \xi) \\ &= \sum_{i=0}^{N-1} \int \left(\frac{\eta(i)}{\lambda_1(t, i/N)} - 1 \right) \left(\frac{1}{2} \frac{\partial^2}{\partial \theta^2} \lambda_1(t, i/N) + \tilde{\beta}_1(\lambda_1(t, i/N), \lambda_2(t, i/N)) - \right. \\ &\quad \left. \tilde{\delta}_1(\lambda_1(t, i/N), \lambda_2(t, i/N)) + \tilde{g}(\lambda_1(t, i/N), \lambda_2(t, i/N)) \right) d\mu_t^N(\eta, \xi) \\ &+ \sum_{i=0}^{N-1} \int \left(\frac{\xi(i)}{\lambda_2(t, i/N)} - 1 \right) \left(\frac{1}{2} \frac{\partial^2}{\partial \theta^2} \lambda_2(t, i/N) + \tilde{\beta}_2(\lambda_1(t, i/N), \lambda_2(t, i/N)) - \right. \\ &\quad \left. \tilde{\delta}_2(\lambda_1(t, i/N), \lambda_2(t, i/N)) - \tilde{g}(\lambda_1(t, i/N), \lambda_2(t, i/N)) \right) d\mu_t^N(\eta, \xi). \end{aligned}$$

Since the birth and death rates are not bounded, we have to truncate them with indicators of sets like $A_M = \{\eta(i) + \xi(i) \leq M\}$. To control terms with $\{\eta(i) \geq M\}$ or $\{\xi(i) \geq M\}$, we need the

Lemma 4.3 *Let φ be a function on $\mathbb{N} \times \mathbb{N}$ such that*

$$\lim_{k_1 \rightarrow +\infty} \frac{\varphi(k_1, k_2)}{\delta_1(k_1, k_2)} = 0. \quad (44)$$

Then,

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{x=0}^{N-1} \int_0^T \int \varphi(\eta(x), \xi(x)) \mathbf{1}_{\{\eta(x) > M\}} f_s^N(\eta, \xi) d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) ds = 0. \quad (45)$$

Let φ be a function on $\mathbb{N} \times \mathbb{N}$ such that $\lim_{k_2 \rightarrow +\infty} \frac{\varphi(k_1, k_2)}{\delta_2(k_1, k_2)} = 0$, then

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{x=0}^{N-1} \int_0^T \int \varphi(\eta(x), \xi(x)) \mathbf{1}_{\{\xi(x) > M\}} f_s^N(\eta, \xi) d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) ds = 0. \quad (46)$$

PROOF. We will use a martingale argument. By (44), for all $\varepsilon > 0$ there exists $M_1 \in \mathbb{N}$ such that, for all $M \geq M_1$

$$\varphi(\eta(x), \xi(x)) \mathbf{1}_{\{\eta(x) > M\}} \leq \frac{\varepsilon}{2} \delta_1(\eta(x), \xi(x)) \mathbf{1}_{\{\eta(x) > M\}}.$$

Moreover, by the explicit formulas for β_1 and δ_1 , given in (10), it follows that there exists $C > 0$ such that for $\varepsilon > 0$ and $M > M_1$

$$\varphi(\eta(x), \xi(x)) \mathbf{1}_{\{\eta(x) > M\}} \leq \varepsilon \left[\delta_1(\eta(x), \xi(x)) - \beta_1(\eta(x), \xi(x)) - g(\eta(x), \xi(x)) + C \right]. \quad (47)$$

We have the following centered martingale with respect to the filtration $\mathcal{F}_t = \sigma\{(\eta_s, \xi_s); s \leq t\}$

$$\begin{aligned} M_t^N &= \sum_{x=0}^{N-1} \eta_t(x) - \sum_{x=0}^{N-1} \eta_0(x) - \int_0^t \Omega_N^{\mathcal{R}} \left(\sum_{x=0}^{N-1} \eta_s(x) \right) ds \\ &= \sum_{x=0}^{N-1} \eta_t(x) - \sum_{x=0}^{N-1} \eta_0(x) \\ &\quad + \sum_{x=0}^{N-1} \int_0^t \left[\delta_1(\eta_s(x), \xi_s(x)) - \beta_1(\eta_s(x), \xi_s(x)) - g(\eta_s(x), \xi_s(x)) \right] ds. \end{aligned}$$

Because M_t^N is centered, by the entropy inequality (33) and by (47), we obtain for M large enough and ε small:

$$\begin{aligned} &\frac{1}{N} \sum_{x=0}^{N-1} \int_0^t \int \varphi(\eta(x), \xi(x)) \mathbf{1}_{\{\eta(x) > M\}} f_s^N(\eta, \xi) d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) ds \\ &\leq \frac{\varepsilon}{N} \sum_{x=0}^{N-1} \int_0^t \int \left[\delta_1(\eta(x), \xi(x)) - \beta_1(\eta(x), \xi(x)) - g(\eta(x), \xi(x)) + C \right] d\mu_s^N(\eta, \xi) ds. \\ &\leq t \varepsilon \frac{1}{N} \sum_{x=0}^{N-1} \left(C - \int \eta(x) f_t^N(\eta, \xi) d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) + \int \eta(x) f^N(\eta, \xi) d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) \right) \\ &\leq \varepsilon C_t, \end{aligned}$$

therefore (45) follows.

The computation to prove (46) is quite different; for all $\varepsilon > 0$ we have

$$\begin{aligned} & \varepsilon \sum_{x=0}^{N-1} \left[\beta_2(\eta(x), \xi(x)) + \lambda \mathbf{1}_{\{\eta(x)>0, \xi(x)=0\}} \left(\sum_{|y-x|=1} \xi(y) \right) + \phi \mathbf{1}_{\{\eta(x)>M\}} \xi(x) \right] \\ & \leq \frac{\varepsilon}{2} \sum_{x=0}^{N-1} \left[\delta_2(\eta(x), \xi(x)) + 2C \right], \end{aligned}$$

it follows that there exists $C > 0$ such that for all $\varepsilon > 0$,

$$\sum_{x=0}^{N-1} \left[\frac{\varepsilon}{2} \delta_2(\eta(x), \xi(x)) - \varepsilon \beta_2(\eta(x), \xi(x)) + \varepsilon g(\eta(x), \xi(x)) + \varepsilon C \right] > 0.$$

Finally we obtain the result by the following inequality

$$\sum_{x=0}^{N-1} \varphi(\eta(x), \xi(x)) \mathbf{1}_{\{\xi(x)>M\}} \leq \varepsilon \sum_{x=0}^{N-1} \left[\delta_2(\eta(x), \xi(x)) - \beta_2(\eta(x), \xi(x)) - g(\eta(x), \xi(x)) + C \right]. \quad \square$$

Let us now integrate (41), putting things together, and removing the negative terms:

$$\begin{aligned} \frac{1}{N} \mathbb{H} \left[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N \right] & \leq \frac{1}{N} \mathbb{H} \left[\mu^N | \nu_{m_1(\cdot)}^N \times \nu_{m_2(\cdot)}^N \right] + F(M, N, T) + o\left(\frac{1}{N}\right) \\ & + \frac{1}{N} \sum_{i=0}^{N-1} \int_0^t \int \left\{ \sum_{i=1}^7 T_i \right\} d\mu_s^N(\eta, \xi) ds. \end{aligned} \quad (48)$$

In order to simplify the expression of T_i , ($i = 1 \dots, 7$), we set:

$$\begin{aligned} \beta_{1,M}(\eta(i), \xi(i)) &= \beta_1(\eta(i), \xi(i)) \mathbf{1}_{\{\eta(i) \leq M, \xi(i) \leq M\}} \\ \varphi_{1,M}(\eta(i), \xi(i)) &= \eta(i) \beta_1(\eta(i) - 1, \xi(i)) \mathbf{1}_{\{\eta(i) \leq M+1, \xi(i) \leq M\}}, \\ T_1 &= \frac{1}{\lambda_1(s, i/N)} \varphi_{1,M}(\eta(i), \xi(i)) - \beta_{1,M}(\eta(i), \xi(i)) - \\ & \quad \left(\frac{\eta(i)}{\lambda_1(s, i/N)} - 1 \right) \tilde{\beta}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)). \end{aligned}$$

$$\begin{aligned} \delta_{1,M}(\eta(i), \xi(i)) &= \delta_1(\eta(i), \xi(i)) \mathbf{1}_{\{\eta(i) \leq M, \xi(i) \leq M\}} \\ \phi_{1,M}(\eta(i), \xi(i)) &= \frac{1}{\eta(i) + 1} \delta_1(\eta(i) + 1, \xi(i)) \mathbf{1}_{\{\eta(i) \leq M-1, \xi(i) \leq M\}}, \end{aligned}$$

and

$$\begin{aligned} T_2 &= \lambda_1(s, i/N) \phi_{1,M}(\eta(i), \xi(i)) - \delta_{1,M}(\eta(i), \xi(i)) + \\ & \quad \left(\frac{\eta(i)}{\lambda_1(s, i/N)} - 1 \right) \tilde{\delta}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)), \end{aligned}$$

$$\begin{aligned} \beta_{2,M}(\eta(i), \xi(i)) &= \beta_2(\eta(i), \xi(i)) \mathbf{1}_{\{\eta(i) \leq M, \xi(i) \leq M\}} \\ \varphi_{2,M}(\eta(i), \xi(i)) &= \xi(i) \beta_2(\eta(i), \xi(i) - 1) \mathbf{1}_{\{\eta(i) \leq M, \xi(i) \leq M+1\}}, \end{aligned}$$

and

$$T_3 = \frac{1}{\lambda_2(s, i/N)} \varphi_{2,M}(\eta(i), \xi(i)) - \beta_{2,M}(\eta(i), \xi(i)) - \left(\frac{\xi(i)}{\lambda_2(s, i/N)} - 1 \right) \tilde{\beta}_2(\lambda_1(s, i/N), \lambda_2(s, i/N)),$$

$$\delta_{2,M}(\eta(i), \xi(i)) = \delta_2(\eta(i), \xi(i)) \mathbf{1}_{\{\eta(i) \leq M, \xi(i) \leq M\}}$$

$$\phi_{2,M}(\eta(i), \xi(i)) = \frac{1}{\xi(i) + 1} \delta_2(\eta(i), \xi(i) + 1) \mathbf{1}_{\{\eta(i) \leq M, \xi(i) \leq M-1\}},$$

and

$$T_4 = \lambda_2(s, i/N) \phi_{2,M}(\eta(i), \xi(i)) - \delta_{2,M}(\eta(i), \xi(i)) + \left(\frac{\xi(i)}{\lambda_2(s, i/N)} - 1 \right) \tilde{\delta}_2(\lambda_1(s, i/N), \lambda_2(s, i/N)),$$

$$e_{1,M}(\eta(i), \xi(i)) = \phi \xi(i) \mathbf{1}_{\{\eta(i) > 0, \xi(i) \leq M\}}$$

$$E_{1,M}(\eta(i), \xi(i)) = \phi \frac{\xi(i)(\xi(i) - 1)}{\eta(i) + 1} \mathbf{1}_{\{\xi(i) \leq M+1\}},$$

and

$$T_5 = \frac{\lambda_1(s, i/N)}{\lambda_2(s, i/N)} E_{1,M}(\eta(i), \xi(i)) - e_{1,M}(\eta(i), \xi(i)) + \left(\frac{\eta(i)}{\lambda_1(s, i/N)} - \frac{\xi(i)}{\lambda_2(s, i/N)} \right) \times \tilde{e}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)),$$

$$e_{2,M}(\eta(i), \xi(i)) = \lambda \mathbf{1}_{\{\eta(i) > 0, \xi(i) = 0\}} \left(\sum_{j \in \mathbb{T}_N} p(j, i) \xi(j) \mathbf{1}_{\{\xi(j) \leq M\}} \right),$$

$$E_{2,M}(\eta(i), \xi(i)) = \frac{\lambda \mathbf{1}_{\{\xi(i) = 1\}}}{\eta(i) + 1} \left(\sum_{j \in \mathbb{T}_N} p(j, i) \xi(j) \mathbf{1}_{\{\xi(j) \leq M\}} \right),$$

and

$$T_6 = \frac{\lambda_1(s, i/N)}{\lambda_2(s, i/N)} E_{2,M}(\eta(i), \xi(i)) - e_{2,M}(\eta(i), \xi(i)) + \left(\frac{\eta(i)}{\lambda_1(s, i/N)} - \frac{\xi(i)}{\lambda_2(s, i/N)} \right) \times \tilde{e}_{2,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)).$$

$$r_M(\eta(i), \xi(i)) = \xi(i) \mathbf{1}_{\{\xi(i) \leq M, \eta(i) \leq M\}}$$

$$R_M(\eta(i), \xi(i)) = \eta(i) \mathbf{1}_{\{\xi(i) \leq M-1, \eta(i) \leq M+1\}},$$

and

$$T_7 = \frac{\lambda_2(s, i/N)}{\lambda_1(s, i/N)} R_M(\eta(i), \xi(i)) - r_M(\eta(i), \xi(i)) - \left(\frac{\eta(i)}{\lambda_1(s, i/N)} - \frac{\xi(i)}{\lambda_2(s, i/N)} \right) \times \tilde{r}_M(\lambda_1(s, i/N), \lambda_2(s, i/N)),$$

Then $F(M, N, T)$ contains all terms with $\mathbf{1}_{\{\eta(x) > M\}}$ and $\mathbf{1}_{\{\xi(x) > M\}}$:

$$\begin{aligned}
F(M, N, T) = & \frac{1}{N} \sum_{i=0}^{N-1} \int_0^T \int f_s^N(\eta, \xi) \left[\right. \\
& \left(\frac{1}{\lambda_1(s, i/N)} \varphi_1(\eta(i), \xi(i)) + \lambda_1(s, i/N) \phi_1(\eta(i), \xi(i)) \right) \times \left(\mathbf{1}_{\{\eta(i) \geq M\}} + \mathbf{1}_{\{\xi(i) \geq M\}} \right) \\
& + \left(\frac{1}{\lambda_2(s, i/N)} \varphi_2(\eta(i), \xi(i)) + \lambda_2(s, i/N) \phi_2(\eta(i), \xi(i)) \right) \times \left(\mathbf{1}_{\{\eta(i) \geq M\}} + \mathbf{1}_{\{\xi(i) \geq M\}} \right) \\
& + \frac{\lambda_2(s, i/N)}{\lambda_1(s, i/N)} \eta(i) \left(\mathbf{1}_{\{\eta(i) \geq M\}} + \mathbf{1}_{\{\xi(i) \geq M\}} \right) + \frac{\eta(i)}{\lambda_1(s, i/N)} \tilde{\delta}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) \\
& + \phi \frac{\lambda_1(s, i/N)}{\lambda_2(s, i/N)} \frac{\xi(i)(\xi(i) - 1)}{\eta(i) + 1} \mathbf{1}_{\{\xi(i) \geq M\}} + \frac{\xi(i)}{\lambda_2(s, i/N)} \tilde{\delta}_{2,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) \\
& + \frac{\lambda_1(s, i/N)}{\lambda_2(s, i/N)} \frac{\lambda}{\eta(i) + 1} \mathbf{1}_{\{\xi(i)=1\}} \left(\sum_{|j-i|=1} \xi(j) \mathbf{1}_{\{\xi(j) > M\}} \right) + \\
& \frac{\eta(i)}{\lambda_1(s, i/N)} \tilde{r}_M(\lambda_1(s, i/N), \lambda_2(s, i/N)) \\
& + \tilde{\beta}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) + \tilde{\beta}_{2,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) \\
& + \frac{\xi(i)}{\lambda_2(s, i/N)} \left(\tilde{e}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) + \right. \\
& \left. \tilde{e}_{2,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) \right) \left. \right] d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) ds,
\end{aligned}$$

and for $k = 1, 2$

$$\begin{aligned}
\tilde{\beta}_{k,M}(a_1, a_2) &= \int \left[\beta_k(\eta, \xi) - \beta_{k,M}(\eta, \xi) \right] d(\nu_{a_1}^N \times \nu_{a_2}^N)(\eta, \xi) \\
\tilde{\delta}_{k,M}(a_1, a_2) &= \int \left[\delta_k(\eta, \xi) - \delta_{k,M}(\eta, \xi) \right] d(\nu_{a_1}^N \times \nu_{a_2}^N)(\eta, \xi) \\
\tilde{e}_{k,M}(a_1, a_2) &= \int \left[e_k(\eta, \xi) - e_{k,M}(\eta, \xi) \right] d(\nu_{a_1}^N \times \nu_{a_2}^N)(\eta, \xi) \\
\tilde{r}_M(a_1, a_2) &= \int \left[r(\eta, \xi) - r_M(\eta, \xi) \right] d(\nu_{a_1}^N \times \nu_{a_2}^N)(\eta, \xi).
\end{aligned}$$

To control the term $F(M, N, T)$ we use lemma 4.3 to obtain

$$\lim_{M \rightarrow +\infty} \limsup_{N \rightarrow +\infty} F(M, N, T) = 0. \quad (49)$$

For the rest of the paper we need the following result due to Perrut (2000). For each bounded function h on $\mathbb{N} \times \mathbb{N}$ and for all x_1, x_2, y_1, y_2 in \mathbb{R}^+ , we set

$$(\Gamma h)(x_1, x_2, y_1, y_2) = \tilde{h}(x_1, x_2) - \tilde{h}(y_1, y_2) - \frac{d\tilde{h}}{dx_1}(y_1, y_2)(x_1 - y_1) - \frac{d\tilde{h}}{dx_2}(y_1, y_2)(x_2 - y_2) \quad (50)$$

Lemma 4.4 (Perrut, 2000) Let $h(.,.)$ be a bounded function on $\mathbb{N} \times \mathbb{N}$, $\rho_1(.)$ and $\rho_2(.)$ be two positive bounded functions on $[0, 1]$ and J be a continuous function on \mathbb{R}^2 . Then there exists $\gamma_0 > 0$, such that, for all $\gamma \leq \gamma_0$,

$$\frac{1}{N} \sum_{i=0}^{N-1} \int J(\rho_1(i/N), \rho_2(i/N)) (\Gamma h)(\eta^k(i), \xi^k(i), \rho_1(i/N), \rho_2(i/N)) \times \\ f_t^N(\eta, \xi) d(\nu_\rho^N \times \nu_\rho^N)(\eta, \xi) \leq \frac{1}{\gamma N} H \left[\mu_t^N | \nu_{\rho_1(\cdot)}^N \times \nu_{\rho_2(\cdot)}^N \right] + R_N^t(k, \gamma),$$

with $\limsup_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} R_N^t(k, \gamma) \leq 0$.

Lemma 4.5 For $k = 1, 2$ we have

$$\begin{aligned} a) \quad & \frac{1}{\lambda_k} \tilde{\varphi}_{k,M}(x_1, x_2) - \tilde{\beta}_{k,M}(x_1, x_2) - \left(\frac{x_k}{\lambda_k} - 1 \right) \tilde{\beta}_{k,M}(\lambda_1, \lambda_2) = \\ & \frac{1}{\lambda_k} \Gamma \varphi_{k,M}(x_1, x_2, \lambda_1, \lambda_2) - \Gamma \beta_{k,M}(x_1, x_2, \lambda_1, \lambda_2) \\ b) \quad & \lambda_k \tilde{\phi}_{k,M}(x_1, x_2) - \tilde{\delta}_{k,M}(x_1, x_2) + \left(\frac{x_k}{\lambda_k} - 1 \right) \tilde{\delta}_{k,M}(\lambda_1, \lambda_2) = \\ & \frac{1}{\lambda_k} \Gamma \phi_{k,M}(x_1, x_2, \lambda_1, \lambda_2) - \Gamma \beta_{k,M}(x_1, x_2, \lambda_1, \lambda_2) \\ c) \quad & \frac{\lambda_1}{\lambda_2} \tilde{E}_{k,M}(x_1, x_2) - \tilde{e}_{k,M}(x_1, x_2) - \left(\frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2} \right) \tilde{e}_{k,M}(\lambda_1, \lambda_2) = \\ & \frac{\lambda_1}{\lambda_2} \Gamma E_{k,M}(x_1, x_2, \lambda_1, \lambda_2) - \Gamma e_{k,M}(x_1, x_2, \lambda_1, \lambda_2) \\ d) \quad & \frac{\lambda_2}{\lambda_1} \tilde{R}_M(x_1, x_2) - \tilde{r}_M(x_1, x_2) + \left(\frac{x_1}{\lambda_1} - \frac{x_2}{\lambda_2} \right) \tilde{r}_M(\lambda_1, \lambda_2) = \\ & \frac{\lambda_2}{\lambda_1} \Gamma R_M(x_1, x_2, \lambda_1, \lambda_2) - \Gamma r_M(x_1, x_2, \lambda_1, \lambda_2) \end{aligned}$$

PROOF. We use substitution rule (32) and the formula (50) of Γ . We need only to remark that for $k = 1, 2$

$$\begin{aligned} \tilde{\varphi}_{k,M}(\lambda_1, \lambda_2) &= \lambda_k \tilde{\beta}_{k,M}(\lambda_1, \lambda_2), \quad \tilde{\phi}_{k,M}(\lambda_1, \lambda_2) = \frac{1}{\lambda_k} \tilde{\delta}_{k,M}(\lambda_1, \lambda_2). \\ \tilde{E}_{k,M}(\lambda_1, \lambda_2) &= \frac{\lambda_2}{\lambda_1} \tilde{e}_{k,M}(\lambda_1, \lambda_2), \quad \tilde{R}_M(\lambda_1, \lambda_2) = \frac{\lambda_1}{\lambda_2} \tilde{r}_M(\lambda_1, \lambda_2). \end{aligned}$$

□

All terms of the upper bound (48) of the relative entropy are evaluated in the same way, it is enough to compute for example the first one. We shall replace the local functions $\varphi_M, \beta_M, \eta(\cdot)$ and $\xi(\cdot)$ by functions of the empirical density of the particles in boxes of size $2k + 1$, with k going to infinity after N . This is possible thanks to the one block estimate, that is proposition 4.1.

$$\begin{aligned} T_1 &= \frac{1}{N} \sum_{i=0}^{N-1} \int_0^t \int \left[\frac{\varphi_{1,M}(\eta(i), \xi(i))}{\lambda_1(s, i/N)} - \beta_{1,M}(\eta(i), \xi(i)) - \right. \\ & \quad \left. \left(\frac{\eta(i)}{\lambda_1(s, i/N)} - 1 \right) \tilde{\beta}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) \right] d\mu_s^N(\eta, \xi) ds \\ &\leq \frac{1}{N} \sum_{i=0}^{N-1} \int_0^t \int \left[\frac{\tilde{\varphi}_{1,M}(\eta^k(i), \xi^k(i))}{\lambda_1(s, i/N)} - \tilde{\beta}_{1,M}(\eta^k(i), \xi^k(i)) - \right. \\ & \quad \left. \left(\frac{\eta^k(i)}{\lambda(s, i/N)} - 1 \right) \tilde{\beta}_{1,M}(\lambda_1(s, i/N), \lambda_2(s, i/N)) + r_N^s(M, k) \right] d\mu_s^N(\eta, \xi) ds, \end{aligned}$$

where $\limsup_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} r_N^t(M, k) \leq 0$. By a) of lemma 4.5, we have

$$T_1 \leq \frac{1}{N} \sum_{i=0}^{N-1} \int_0^t \int \left[\frac{1}{\lambda_1(s, i/N)} (\Gamma \varphi_{1,M}) (\eta^k(i), \xi^k(i), \lambda_1(s, i/N), \lambda_2(s, i/N)) - \right. \\ \left. (\Gamma \beta_{1,M}) (\eta^k(i), \xi^k(i), \lambda_1(s, i/N), \lambda_2(s, i/N)) \right] d\mu_s^N(\eta, \xi) ds.$$

By lemma 4.4, there exists $\gamma_0 > 0$, such that for all $\gamma \leq \gamma_0$

$$T_1 \leq \frac{2}{\gamma N} \int_0^t H \left[\mu_s^N | \nu_{\lambda_1(s, \cdot)}^N \times \nu_{\lambda_2(s, \cdot)}^N \right] ds + R_N^t(k, \gamma) + r_N^t(M, k),$$

where $\limsup_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} R_N^t(k, \gamma) \leq 0$. Then

$$\frac{1}{N} H \left[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N \right] \leq \frac{1}{N} H \left[\mu^N | \nu_{m_1(\cdot)}^N \times \nu_{m_2(\cdot)}^N \right] + r_N^t(M, k) + R_N^t(k, \gamma) + \\ F(M, N, T) + \frac{14}{\gamma N} \int_0^t H \left[\mu_t^N | \nu_{\lambda_1(t, \cdot)}^N \times \nu_{\lambda_2(t, \cdot)}^N \right] ds + o(1/N) \quad (51)$$

Finally, hypothesis (15) and Gronwall lemma imply (39), which ends the proof. \square

4.3. Proof of Theorem 4 (extension to infinite volume) To extend Theorem 3 to infinite volume, that is to all space \mathbb{Z} , we follow the same strategy as in Perrut (2000), and in Landim and Yau, (1995), we make a coupling between two processes: the first one $(\eta_t^1, \xi_t^1)_{t \geq 0}$ on \mathbb{Z} with μ^N as initial distribution and the second one $(\eta_t^2, \xi_t^2)_{t \geq 0}$ on $\mathbb{T}_{CN} = \{-CN, \dots, CN\}$ with μ^N restricted to \mathbb{T}_{CN} as initial distribution. We will prove that when N goes to infinity and C is large the “difference” between those two processes is small in a sense to be specified later.

To couple $(\eta_t^1, \xi_t^1)_{t \geq 0}$ and $(\eta_t^2, \xi_t^2)_{t \geq 0}$ we distinguish between two types of particles: the coupled ones and the non-coupled ones. More precisely, at site x , the $\eta_t^1(x)$ -particles are divided into $\eta_t^*(x)$ and $\eta_t^{1*}(x)$. The $\eta_t^*(x)$ -particles are associated to particles of $\eta_t^2(x)$, these couples of particles move together. All the other particles stay single. Initially $\eta_0^*(x) = \eta_0^1(x) \wedge \eta_0^2(x)$ for all $x \in \mathbb{T}_{CN}$. We set $\eta_0^1(x) = \eta_0^*(x) + \eta_0^{1*}(x)$, $\eta_0^2(x) = \eta_0^*(x) + \eta_0^{2*}(x)$ and do the same for ξ_0^1 and ξ_0^2 .

The diffusion part of the coupled generator $\overline{\Omega}_N$, is denoted by $\overline{\Omega}_N^{\mathcal{D}}$, where

$$\overline{\Omega}_N^{\mathcal{D}} = \overline{\Omega}_N^{\mathcal{D},1} + \overline{\Omega}_N^{\mathcal{D},2},$$

and $\overline{\Omega}_N^{\mathcal{D},1}$ describes at sites $|x| < CN$ the evolution by:

$$\begin{aligned} \overline{\Omega}_N^{\mathcal{D},1} f(\eta^*, \eta^{*1}, \eta^{*2}) &= \sum_{\substack{|x| < CN \\ y \in \mathbb{T}_{CN}}} p(x, y) \eta^*(x) \left[f((\eta^*)^{x,y}, \eta^{*1}, \eta^{*2}) - f(\eta^*, \eta^{*1}, \eta^{*2}) \right] \\ &+ \sum_{\substack{|x| < CN \\ y \in \mathbb{T}_{CN}}} p(x, y) \eta^{1*}(x) \wedge \eta^{*2}(x) \left[f(\eta^*, (\eta^{*1})^{x,y}, (\eta^{*2})^{x,y}) - f(\eta^*, \eta^{*1}, \eta^{*2}) \right] \\ &+ \sum_{\substack{|x| < CN \\ y \in \mathbb{T}_{CN}}} p(x, y) \left(\eta^{1*}(x) - \eta^{*2}(x) \right)^+ \left[f(\eta^*, (\eta^{*1})^{x,y}, \eta^{*2}) - f(\eta^*, \eta^{*1}, \eta^{*2}) \right] \\ &+ \sum_{\substack{|x| < CN \\ y \in \mathbb{T}_{CN}}} p(x, y) \left(\eta^{*2}(x) - \eta^{*1}(x) \right)^+ \left[f(\eta^*, \eta^{*1}, (\eta^{*2})^{x,y}) - f(\eta^*, \eta^{*1}, \eta^{*2}) \right]. \end{aligned}$$

At site $x = CN$, the particles of the two processes jump outside $\{-CN, \dots, CN\}$ independently. Those of η^1 arrive at $CN + 1$ and the others at $-CN$. The coupled generator $\overline{\Omega}_N^{\mathcal{D},2}$ of infected individuals ξ evolves according to the same rules.

The reaction part of the coupled generator $\overline{\Omega}_N$ is denoted by $\overline{\Omega}_N^{\mathcal{R}}$ and defined for all cylinder function as the sum of $\overline{\Omega}_N^{\mathcal{R},i}$, $i = 1, \dots, 5$. Let $\overline{\Omega}_N^{\mathcal{R},1}$ be the coupled generator of birth and death of healthy individuals described at sites $|x| < CN$ by:

At rate $\beta_1(\eta^1(x), \xi^1(x)) \wedge \beta_1(\eta^2(x), \xi^2(x))$ (respectively, $\delta_1(\eta^1(x), \xi^1(x)) \wedge \delta_1(\eta^2(x), \xi^2(x))$) two coupled particles are created (respectively, removed), at rate $(\beta_1(\eta^1(x), \xi^1(x)) - \beta_1(\eta^2(x), \xi^2(x)))^+$ (respectively, $(\delta_1(\eta^2(x), \xi^2(x)) - \delta_1(\eta^1(x), \xi^1(x)))^+$) a particle of η^{1*} is created (respectively, removed), and at rate $(\beta_1(\eta^2(x), \xi^2(x)) - \beta_1(\eta^1(x), \xi^1(x)))^+$ (respectively, $(\delta_1(\eta^2(x), \xi^2(x)) - \delta_1(\eta^1(x), \xi^1(x)))^+$) a particle of η^{2*} is created (respectively, removed). In a symmetric way we define the coupled generator describing birth and death of infected individuals. In the same way, we define the coupled process of recoveries and infection (inside infection, outside infection and recoveries of infected individuals).

We denote by \overline{E}_{μ^N} the expectation of the coupled process $\overline{\Omega}_N$ starting from μ^N . For notational simplicity, we assume that $\alpha_1 + \alpha_2 \leq 1$, and we set

$$\zeta_s^*(x) = \eta_s^{*1}(x) + \eta_s^{*2}(x) + \xi_s^{*1}(x) + \xi_s^{*2}(x).$$

For $x \in \mathbb{T}_{CN}$, since $\zeta(x)$ is constant for the coupled generators for the outside and inside infections and the recoveries, we have:

$$\overline{\Omega}_N^{\mathcal{R}}(\zeta^*(x)) = \overline{\Omega}_N^{\mathcal{R},1}(\zeta^*(x)) \leq (\alpha_1 + \alpha_2)|\eta^1(x) - \eta^2(x)| + (\alpha_1 + \alpha_2)|\xi^1(x) - \xi^2(x)|$$

Thus,

$$\overline{\Omega}_N^{\mathcal{R}}(\zeta_s^*(x)) \leq \zeta_s^*(x). \quad (52)$$

Furthermore, since the death rates are larger than the birth rates it exists a real $c_0 > 0$ such that

$$\overline{\Omega}_N^{\mathcal{R}}(\eta^1(x) + \xi^1(x)) \leq \beta_1(\eta^1(x), \xi^1(x)) - \delta_1(\eta^1(x), \xi^1(x)) + \beta_2(\eta^1(x), \xi^1(x)) - \delta_2(\eta^1(x), \xi^1(x)) \leq c_0,$$

and since $\frac{d}{dt} \overline{E}_{\mu^N} [f(\eta_t, \xi_t)] = \overline{E}_{\mu^N} [\overline{\Omega}_N^{\mathcal{R}} f(\eta_t, \xi_t)]$,

$$\overline{E}_{\mu^N} [\overline{\Omega}_N^{\mathcal{R}}(\eta_t^1(x) + \xi_t^1(x))] \leq c_0, \quad \overline{E}_{\mu^N}(\eta_t^1(x) + \xi_t^1(x)) \leq M + t c_0, \quad (53)$$

and

$$\overline{E}_{\mu^N}(\zeta_s^*(x)) \leq 2M + 2t c_0 := K_1. \quad (54)$$

Let $A \in \mathbb{N}$ be fixed. Now we have all the necessary tools to bound above the discrepancy between the two processes in the box $\Lambda_{AN} = \{-NA, \dots, NA\}$. By following the same steps as in Perrut (1999) we prove first by using (52), (53) and (54) that

$$\lim_{C \rightarrow \infty} \lim_{N \rightarrow \infty} \overline{E}_{\mu^N} \left[\frac{1}{N} \sum_{x \in \Lambda_{AN}} \zeta_t^*(x) \right] = 0, \quad (55)$$

and then theorem 4.

Acknowledgment. We thank Ellen Saada for many useful advice and fruitful discussions.

References

- [1] Belhadji, L. and Lanchier, N. (2006). Individual versus cluster recoveries within a spatially structured population. *Ann. Appl. Probab.* **16** 403–422.

- [2] Chen, M.F. (1992). *From Markov chains to non-equilibrium particle systems*. World Scientific, Singapore.
- [3] Durrett, R. (1995). Ten lectures on particle systems. Saint-Flour Lecture Notes, *Lect. Notes Math.* **1608** 97–201.
- [4] Harris, T.E. (1972). Nearest neighbor Markov interaction processes on multidimensional lattices. *Adv. Math.* **9** 66–89.
- [5] Harris, T.E. (1974). Contact interactions on a lattice. *Ann. Probab.* **2** 969–988.
- [6] Kipnis, C. and Landim, C. (1999). *Scaling limits of interacting particle systems*. Springer-Verlag Berlin Heidelberg.
- [7] Liggett, T.M. (1999). *Stochastic interacting systems : contact, voter and exclusion processes*. Berlin Heidelberg New York : Springer.
- [8] Mourragui, M. (1996). Comportement hydrodynamique et entropie relative des processus de sauts, de naissances et de morts. *Ann. Inst. H. Poincaré Probabilités.* **32** 361-385.
- [9] Neuhauser, C. (2001). Mathematical challenges in spatial ecology. *Notices Amer. Math. Soc.* **48** 1304–1314.
- [10] Perrut, A. (2000). Hydrodynamics limits for two-species reaction-diffusion process. *Ann. Appl. Probab.* **10** 163-191.
- [11] Schinazi, R. (2002). On the role of social clusters in the transmission of infectious diseases. *Theor. Popul. Biol.* **61** 163-169.
- [12] Smoller, J. (1983). *Shock waves and reaction-diffusion equations*. Springer, New York.

LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM,
 UMR 6085, CNRS - UNIVERSITÉ DE ROUEN,
 AVENUE DE L'UNIVERSITÉ, BP. 12,
 76801 SAINT ETIENNE DU ROUVRAY, FRANCE.